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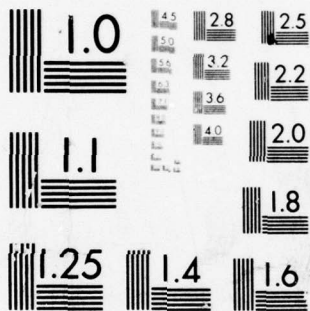
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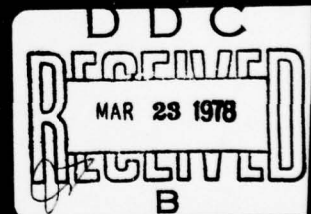
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# G-ORDERED FUNCTIONS IN STATISTICS

by

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## ABSTRACT

This is Part I of a two-part paper which generalizes and extends the work of Hollander, Proschan, and Sethuraman (to appear) for functions decreasing in transposition (DT). In Part I we introduce reflection ordering and G-ordered functions. We develop many preservation properties for these functions. We note that G-ordered functions contain, as a special case, G-monotone functions (Eaton and Perlman (1976)) and we prove a preservation theorem for G-monotone functions under an integral transform. We extend the results of Hollander, Proschan, and Sethuraman (to appear) in the area of stochastic comparisons to a larger class of probability distributions. In Part II we present applications in statistics. In a forthcoming paper we will explore the notion of stochastic G-majorization as an extension of the work done in the area of stochastic majorization by Nevius, Proschan, and Sethuraman (1977).

### 1. Introduction and Summary.

In this two-part paper we introduce and develop the concept of reflection ordering and G-ordered functions. In Part I we explore some of the basic aspects of reflection ordering, G-ordered functions, and the preservation properties of these functions. In Part II we present applications in statistics. Our purpose is to continue the work of unifying the area of stochastic comparisons done by Proschan and Sethuraman (1977), Nevius, Proschan, and Sethuraman (1977), and Hollander, Proschan, and Sethuraman (to appear).

In Section 2 we define reflection groups and enumerate some of their properties. We determine the fundamental regions for any finite reflection group and define for each distinct fundamental region a partial ordering

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on the group. The partial ordering associated with a distinct fundamental region is unique. We present a partial ordering on  $R^n$  with respect to a fundamental region through the ordering defined on the group. The partial ordering on  $R^n$  is useful in applications whereas the ordering on the group unifies the theory and contributes to mathematical elegance.

In Section 3 we define G-ordered functions for any finite reflection group acting on  $R^n$ . We show that the G-ordered property is preserved under mixtures with respect to a positive measure and under composition with respect to a G-invariant measure. We establish that products of nonnegative G-ordered functions are G-ordered. Preservation under composition is particularly useful in further developing the properties of G-ordered functions.

In the context of new work by Eaton and Perlman (1976), we present in Section 4 a preservation theorem for G-monotone functions under an integral transform. It contains as special cases similar theorems for Schur functions of Proschan and Sethuraman (1977) and Hollander, Proschan, and Sethuraman (to appear). We show how our theorem extends results obtained by these authors to a larger class of probability distributions, in particular to the multivariate normal and the multivariate T. We relate the G-majorization ordering introduced by Eaton and Perlman (1976) to reflection ordering and we show that G-monotone functions are special cases of G-ordered functions.

In Section 5 we present a short example in the area of nonparametric statistics. We show that the Wilcoxon signed-rank statistic is a G-ordered function of the signed ranks. We determine conditions under which tests based on the Wilcoxon signed-rank statistic have power functions G-ordered with respect to F.

## 2. Fundamental Regions and Reflection Ordering.

In this section we introduce the notion of reflection ordering. We use the notion of a fundamental region in Euclidean  $n$ -space with respect to a finite reflection group. (See Benson and Grove (1971), pp. 27-33.) Each fundamental region defines a partial ordering, called reflection ordering, on the elements of the group. We present a short summary of the derivation of reflection groups and fundamental regions.

Throughout this paper  $R^n$  denotes Euclidean  $n$ -space. Elements of  $R^n$  are represented by column vectors and the transpose of a vector  $z$  is denoted by  $z'$ . The unit ball in  $R^n$  is denoted by  $S_n$ , i.e.  $S_n = \{x \in R^n: ||x|| = 1\}$ , where  $||x|| = \sqrt{x'x}$  is the usual Euclidean norm.

**Definition 2.1.** Let  $r \in S_n$  and let  $I_n$  be the  $n \times n$  identity matrix. The matrix,  $M_r = I_n - 2rr'$ , is called the reflection defined by  $r$ .

Geometrically,  $M_r$  reflects points across the  $(n-1)$ -dimensional subspace of  $R^n$  perpendicular to  $r$ . Clearly  $M_r = M_{-r} = M_r' = M_r^{-1}$ . In particular, we note that  $M_r \in O(n)$ , the group of  $n \times n$  orthogonal matrices.

**Definition 2.2.** A closed subgroup  $G$  of  $O(n)$  is called a reflection group if there exists a subset  $\Delta_G^*$  of  $S_n$  such that  $G$  is the smallest closed subgroup of  $O(n)$  containing the set of reflections  $\{M_r: r \in \Delta_G^*\}$ .

We call  $\Delta_G^*$  a generating system of  $G$ . A minimal generating system of  $G$  is called a set of fundamental roots of  $G$ .

**Definition 2.3.** The root system of  $G$ , denoted  $\Delta_G$ , is the set  $\{r \in S_n: M_r \in G\}$ .



For any given  $r \in \Delta_G$  we partition  $R^n$  into the following three subsets:

1.  $H_r^+ = \{x \in R^n: r'x > 0\},$
2.  $H_r^- = \{x \in R^n: r'x < 0\},$
3.  $H_r^0 = \{x \in R^n: r'x = 0\}.$

Since  $M_r x = (I - 2rr')x = x - 2rr'x$ , we note that  $M_r x = x$  if and only if  $x \in H_r^0$ . Thus the set  $H_r^0$  is invariant under the transformation defined by the reflection  $M_r$ .

Define the set  $T_G = \{t \in R^n: r't \neq 0 \text{ for each } r \in \Delta_G\}$ . Thus  $T_G$  is the complement of the set  $\bigcap_{r \in \Delta_G} H_r^0$ . When there is no possibility for

ambiguity we will drop the subscript  $G$  in  $T_G$ . For a fixed  $t \in T$ , define the sets:

1.  $\Delta_t^+ = \{r \in \Delta_G: r't > 0\},$
2.  $\Delta_t^- = \{r \in \Delta_G: r't < 0\}.$

Since  $r \in \Delta_G$  if and only if  $-r \in \Delta_G$ ,  $\Delta_t^+$  and  $\Delta_t^-$  partition  $\Delta_G$  into two sets of the same cardinality.

We call  $\Delta_t^+$  the set of  $t$ -positive roots and we note two useful properties relating to positive roots.

1. For every  $g \in G$ ,  $g \Delta_t^+ = \Delta_{gt}^+.$
2. The equality,  $\Delta_{gt}^+ = \Delta_t^+$ , holds if and only if  $g$  is the identity matrix.

For proofs of the above two statements, we refer the reader to Propositions 4.2.2 and 4.2.3 of Benson and Grove (1971).

We partition  $T$  into certain regions, termed fundamental regions, by means of the equivalence relation defined below. The equivalence relation is based on the set of positive roots.

Definition 2.4. Let  $t, s \in T$ . Then  $t$  is equivalent to  $s$  ( $t \sim s$ ) if  $\Delta_t^+ = \Delta_s^+$ .

Definition 2.5. Let  $t \in T$ . The fundamental region  $F$  defined by  $t$  is the set  $\{s \in T: t \sim s\}$ .

It is evident that for  $t, s \in T$ , if  $t$  is equivalent to  $s$ , then  $t$  and  $s$  define the same fundamental region. For any  $t \in T$ ,  $gt$  defines a different fundamental region for each distinct  $g \in G$ . To see this, note that  $t$  is not equivalent to  $gt$  for  $I \neq g \in G$ . This is true since  $\Delta_{gt}^+ = \Delta_t^+$  if and only if  $g = I$  as claimed in statement 2 above. It is consequently of interest to note that the number of distinct fundamental regions and the number of elements of the group  $G$  are equal.

In light of the definition of fundamental regions and the above assertions, one can easily perceive the following properties of any fundamental region  $F$ . (See Benson and Grove (1971), p. 27.)

1.  $F$  is an open set in  $\mathbb{R}^n$ .
2.  $F \cap gF = \emptyset$  if  $g \in G$  is not the identity matrix.
3.  $\mathbb{R}^n = \cup \{g\bar{F}: g \in G\}$ , where  $\bar{F}$  is the closure of  $F$  in  $\mathbb{R}^n$ .

Thus the fundamental regions  $\{gF: g \in G\}$  are the equivalence classes under the equivalence relation presented in Definition 2.4.

For any fixed finite reflection group  $G$ , we define a set of partial orderings on the elements of  $G$ . Each distinct fundamental region in  $\mathbb{R}^n$  defines a unique partial ordering. In order to define the ordering on the group  $G$  we present a partition of  $G$ .

For any fundamental region  $F$ , the set  $\Delta_F^+ \subseteq \Delta_G$  is the set of F-positive roots, i.e.  $\Delta_F^+ = \{r \in \Delta_G: r \cdot t > 0 \text{ for all } t \in F\}$ . Fix a root  $r \in \Delta_G$  and

let  $gF$  be a fundamental region. Then  $r \in \Delta_{gF}^+$  or  $r \in \Delta_{gF}^-$ . For the given fixed  $r$  we partition  $G$  into the sets  $G_r^+$  and  $G_r^-$ , where  $G_r^+$  is the set  $\{g \in G: r \in \Delta_{gF}^+\}$  and  $G_r^-$  is the set  $\{g \in G: r \in \Delta_{gF}^-\}$ . Technically,  $G_r^+$  and  $G_r^-$  depend on the fundamental region  $F$  as well as the root  $r$ . We suppress reference to  $F$  unless ambiguity may result.

**Definition 2.8.** Let  $r \in S_n$  and let  $F$  be a fundamental region in  $R^n$ . Then  $g$  is  $r$ -larger than  $M_r g$ , in symbols,  $g \geq^r M_r g$  if and only if  $g \in G_r^+$ .

Note that if  $g \in G_r^-$ , then  $g$  is  $r^*$ -larger than  $M_{r^*} g$ , where  $r^* = -r$ .

**Definition 2.9.** (Reflection Ordering). Let  $F$  be a fundamental region in  $R^n$  and let  $g_1, g_2 \in G$ . If there exists a sequence  $h_0, h_1, \dots, h_m$  in  $G$  satisfying  $g_1 = h_0 \geq^{r_1} h_1 \geq^{r_2} \dots \geq^{r_m} h_m = g_2$ , where  $r_i \in \Delta_F^+$ ,  $i = 1, 2, \dots, m$ , then  $g_1$  is  $\bar{F}$ -larger than  $g_2$ ; in symbols,  $g_1 \geq^{\bar{F}} g_2$ .

The  $G$ -orbit of a point  $x \in R^n$  is the set  $\{gx: g \in G\}$ .

**Definition 2.10.** Suppose  $x_1, x_2 \in R^n$  and they belong to each other's orbit, i.e.  $x_2 = gx_1$  for some  $g \in G$ . Then there exists  $x \in F$  such that  $x_1 = g_1 x$  and  $x_2 = g_2 x$  for some  $g_1, g_2 \in G$ . We say that  $x_1$  is  $\bar{F}$ -larger than  $x_2$  ( $x_1 \geq^{\bar{F}} x_2$ ) if  $g_1 \geq^{\bar{F}} g_2$ .

Definition 2.10 presents a partial ordering on the  $G$ -orbit of a point  $x \in F$ . We have presented our definition in terms of any fundamental region in  $R^n$ . In most statistical examples the appropriate fundamental region will suggest itself naturally.

To illustrate the preceding concepts let us consider  $G = P_n$ , the group of all  $n!$  permutation matrices. A generating system of  $G$ ,  $\Delta_G^*$ , is the set



$\{r_i: i = 1, 2, \dots, n-1\}$ , where  $r_i' = (0, \dots, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0)$  with  $\frac{1}{\sqrt{2}}$  and  $\frac{1}{\sqrt{2}}$  being the  $i^{\text{th}}$  and  $(i+1)^{\text{st}}$  coordinates of  $r_i$  respectively. A root system of  $G$ ,  $\Delta_G$ , is the set  $\{r_{ij}: 1 \leq i < j \leq n\}$ , where  $r_{ij}' = (0, \dots, -\frac{1}{\sqrt{2}}, 0, \dots, \frac{1}{\sqrt{2}}, 0, \dots, 0)$  with  $-\frac{1}{\sqrt{2}}$  and  $\frac{1}{\sqrt{2}}$  being the  $i^{\text{th}}$  and  $j^{\text{th}}$  coordinates of  $r_{ij}$  respectively. The  $G$ -orbit of a point  $x$  in  $R^n$  is the set of all  $n!$  permutations of the coordinates of  $x$ .

Let us take our fundamental region  $F$  to be the set  $\{x \in R^n: x_1 < x_2 < \dots < x_n\}$ . Since, for  $i < j$ , the  $j^{\text{th}}$  coordinate of  $x \in F$  is larger than the  $i^{\text{th}}$  coordinate,  $\Delta_F^+$  is the set  $\{+r_{ij}: 1 \leq i < j \leq n\}$ . The set  $G_{r_{ij}}^+$ ,  $r_{ij} \in \Delta_G$ , contains any permutation matrix  $g$  such that for  $x \in F$ , the  $i^{\text{th}}$  coordinate of  $gx$  is less than the  $j^{\text{th}}$  coordinate. This is obvious, since  $r_{ij}'gx$  is the  $j^{\text{th}}$  coordinate of  $gx$  less the  $i^{\text{th}}$  coordinate. Now if  $g \in G_{r_{ij}}^+$ , then  $M_{r_{ij}} g \in G_{r_{ij}}^-$ . For  $x \in F$ ,  $gx$   $r_{ij}$  - larger than  $M_{r_{ij}} gx$  means that the  $i^{\text{th}}$  coordinate of  $gx$  is smaller than the  $j^{\text{th}}$  coordinate. The point  $M_{r_{ij}} gx$  is a permutation of the  $i^{\text{th}}$  and  $j^{\text{th}}$  coordinates of  $gx$ . The reader will recognize reflection ordering for the fundamental region  $F$  as the familiar transposition ordering of Hollander, Proschan, and Sethuraman (to appear).

### 3. G-ordered Functions.

In this section we define functions which are isotonic with respect to reflection ordering. We define functions on the group  $G$ , functions on  $R^n$ , and functions on  $R^{2n}$  which have a  $G$ -ordered property. Although the  $G$ -ordered property is essentially a property of functions on the group, it becomes more convenient for theoretical and practical applications to formulate the  $G$ -ordered property for functions on  $R^n$  and  $R^{2n}$ .

$G$ -ordered functions contain as a special case functions decreasing in transposition (DT). (See Hollander, Proschan, and Sethuraman (to appear).)

We establish some basic preservation properties for  $G$ -ordered functions. For example, we show that mixtures and compositions of  $G$ -ordered functions are  $G$ -ordered functions. We note that the  $G$ -ordered property is preserved under products of nonnegative  $G$ -ordered functions.

**Definition 3.1.** A function  $f$  from  $G$  into  $R^1$  is  $G$ -ordered with respect to  $F$  if  $g_1 \geq^{\bar{F}} g_2$  implies  $f(g_1) \geq f(g_2)$ , for  $g_1, g_2 \in G$ .

**Definition 3.2.** Let  $X$  be a subset of  $R^n$ . It is said to be  $G$ -invariant if  $gX \subseteq X$  for all  $g \in G$ .

Unless otherwise specified,  $G$  will be a fixed finite reflection group acting on  $R^n$ , although some of the results are true for arbitrary subgroups of  $O(n)$ . The sets  $\Lambda$  and  $X$ , with or without subscripts, will denote  $G$ -invariant subsets of  $R^n$ .

**Definition 3.3.** A function  $f$  from  $X$  into  $R^1$  is  $G$ -ordered with respect to  $F$  on  $X$  if for every  $x \in \bar{F} \cap X$  and for every pair  $g_1, g_2 \in G$  such that  $g_1 \geq^{\bar{F}} g_2$ , we have  $f(g_1 x) \geq f(g_2 x)$ .

For each  $x$  in some fundamental region  $F$  define the function  $h_x(g) = f(gx)$ . Note that  $h_x$  is  $G$ -ordered with respect to  $F$  on  $G$  if  $f$  is  $G$ -ordered with respect to  $F$  on  $X$ .

**Definition 3.4.** A function  $K$  from  $\Lambda \times X$  into  $R^1$  is  $G$ -ordered if the following two conditions hold.

- i.  $K(g\lambda, gx) = K(\lambda, x)$  for all  $g \in G$ .
- ii. For every fundamental region  $F$  when  $\lambda \in \bar{F} \cap \Lambda$ ,  $x \in \bar{F} \cap X$ , and  $g_1 \geq^{\bar{F}} g_2$ , then  $K(\lambda, g_1 x) \geq K(\lambda, g_2 x)$ .



**Remark 3.5.** The reader should note that condition (i) above can be replaced by:

i\*.  $K(M_r \lambda, M_r x) = K(\lambda, x)$  for all  $r$  in a set of fundamental roots.

**Lemma 3.6.** Let  $K(\lambda, x) = K(g\lambda, g x)$  for all  $g \in G$ . Define

(a)  $\tilde{K}(x, \lambda) = K(\lambda, x)$  for  $\lambda \in \Lambda, x \in X$ .

(b)  $f_\lambda(x) = K(\lambda, x)$  for  $\lambda \in \Lambda, x \in X$ .

(c)  $h_{\lambda, x}(g) = K(\lambda, gx)$  for  $\lambda \in \tilde{g}\bar{F} \cap \Lambda$ , for  $x \in \tilde{g}\bar{F} \cap X$ , for all  $g \in G$ ,

and for some  $\tilde{g} \in G$ . Then the following are equivalent:

(1)  $K$  is  $G$ -ordered on  $\Lambda \times X$ .

(2)  $\tilde{K}$  is  $G$ -ordered on  $X \times \Lambda$ .

(3)  $f_\lambda$  is  $G$ -ordered with respect to  $F$  on  $X$  for each  $\lambda \in \bar{F} \cap \Lambda$ .

(4)  $h_{\lambda, x}$  is  $G$ -ordered with respect to  $\tilde{g}\bar{F}$  on  $G$  for each  $\lambda \in \tilde{g}\bar{F} \cap \Lambda$  and each  $x \in \tilde{g}\bar{F} \cap X$ .

The equivalence follows directly from the definitions of  $G$ -ordered functions on  $G$ , on  $X$ , and on  $\Lambda \times X$ . ||

We now present some preservation properties for  $G$ -ordered functions. The proofs of Propositions 3.7, 3.8, and 3.9 below parallel the proofs of corresponding results in Hollander, Proschan, and Sethuraman (to appear), so we omit them.

**Proposition 3.7.** Let  $K(\lambda, x)$  be  $G$ -ordered on  $\Lambda \times X$  and let  $f$  and  $h$  be nonnegative  $G$ -invariant functions on  $\Lambda$  and  $X$  respectively. Then  $f(\lambda) K(\lambda, x) h(x)$  is  $G$ -ordered on  $\Lambda \times X$ .

**Proposition 3.8.** Let  $(\Omega, F, \nu)$  be a measure space. Suppose  $K_\omega(\lambda, x)$  is  $G$ -ordered on  $\Lambda \times X$  for each  $\omega \in \Omega$ , and suppose that for all  $(\lambda, x) \in \Lambda \times X$ ,  $K_\omega(\lambda, x) \in L^1(\Omega, F, \nu)$ . Then  $\int_\Omega K_\omega(\lambda, x) d\nu(\omega)$  is  $G$ -ordered on  $\Lambda \times X$ .

A similar result for mixtures holds for functions G-ordered with respect to F on G and on X.

Consider any probability density  $Q(\lambda, x)$  defined by  $Q(\lambda, x) = c(\lambda) h(x) \exp(\sum_{i=1}^k Q_i(\lambda, x))$ . Using Proposition 3.7, Proposition 3.8 with the counting measure, and the fact that increasing functions of G-ordered functions are G-ordered, we can show that  $Q(\lambda, x)$  is G-ordered if  $c$  and  $h$  are G-invariant and  $Q_i$ ,  $i = 1, 2, \dots, k$ , is G-ordered. Note that densities belonging to the multivariate exponential family are special cases of this form.

Note that if  $K(\lambda, x)$  is G-ordered on  $\Lambda \times X$ , then  $K(\lambda, x)$  is G-ordered on  $\Lambda^* \times X^*$ , where  $\Lambda^*$  and  $X^*$  are any G-invariant subsets of  $\Lambda$  and  $X$  respectively. Thus if  $K(\lambda, x)$ , a G-ordered function on  $\Lambda \times X$ , is the density of a random variable  $X$  and  $u$  is a G-invariant function on  $X$ , the conditional density of  $X$  given  $u(X) = u_0$ ,  $K_{u_0}(\lambda, x)$  is G-ordered on  $\Lambda \times X_0$ , where  $X_0 = \{x \in X: u(x) = u_0\}$ .

**Proposition 3.9.** The product of nonnegative G-ordered functions is G-ordered.

**Definition 3.10.** A measure  $\mu$  on  $X$  is G-invariant if  $\mu(A \cap X) = \mu(gA \cap X)$  for any  $g \in G$  and any Borel set  $A$  in  $R^n$ .

**Theorem 3.11.** Let  $K_1$  be G-ordered on  $X_1 \times X$  and let  $K_2$  be G-ordered on  $X \times X_2$ . Let  $K(x, z) = \int_X K_1(x, y) K_2(y, z) d\mu(y)$ , where  $\mu$  is a G-invariant measure. Then  $K$  is G-ordered on  $X_1 \times X_2$ .

**Proof.** (i). First we show  $K(gx, gz) = K(x, z)$  for all  $g \in G$ . Let  $g \in G$ . Then

$$\begin{aligned} K(gx, gz) &= \int_X K_1(gx, y) K_2(y, gz) d\mu(y) = \int_X K_1(gx, gy) K_2(gy, gz) d\mu(gy) \\ &= \int_X K_1(x, y) K_2(y, z) d\mu(y) \text{ [using the G-invariance of } \mu] \\ &= K(x, z), \text{ as desired.} \end{aligned}$$

(ii). Now let  $x \in \bar{F} \cap X_1$  and  $z \in \bar{F} \cap X_2$ . We need to show that  $g_1 \geq^{\bar{F}} g_2$  implies  $K(x, g_1 z) \geq K(x, g_2 z)$ . It suffices to show that  $K(x, z) \geq$

$K(x, M_r z)$  for some  $r \in \Delta_F^+$ . Let  $r \in \Delta_F^+$ , then  $K(x, z) - K(x, M_r z)$

$$\begin{aligned} &= \int K_1(x, y) [K_2(y, z) - K_2(y, M_r z)] d\mu(y) \\ &= \int_{H_r^+} K_1(x, y) [K_2(y, z) - K_2(y, M_r z)] d\mu(y) \\ &\quad + \int_{H_r^-} K_1(x, y) [K_2(y, z) - K_2(y, M_r z)] d\mu(y) \\ &\quad + \int_{H_r^0} K_1(x, y) [K_2(y, z) - K_2(y, M_r z)] d\mu(y). \end{aligned}$$

Since  $K_2(y, z) - K_2(y, M_r z) = 0$  for all  $y \in H_r^0$ , we drop the third integral above. We use the transformation  $y = M_r u$  in the second integral above and by the G-invariance of  $\mu$  we conclude that the second integral is equal to:

$$\int_{H_r^+} K_1(x, M_r u) [K_2(u, M_r z) - K_2(u, z)] d\mu(u).$$

We now combine the first and second integral and factor the integrand to obtain that  $K(x, z) - K(x, M_r z) = \int_{H_r^+} [K_1(x, y) - K_1(x, M_r y)] [K_2(y, z) - K_2(y, M_r z)] d\mu(y)$ . Both factors are nonnegative in the region  $y \in H_r^+$ , so  $K(x, z) - K(x, M_r z) \geq 0$ , as desired. ||

**Corollary 3.12.** Let  $K(\lambda, x)$  be G-ordered on  $\Lambda \times X$  and let  $f(x)$  be G-ordered with respect to F on X. Let  $h(\lambda) = \int K(\lambda, x) f(x) d\mu(x)$ , where  $\mu$  is a G-invariant measure on X. Then  $h(\lambda)$  is G-ordered with respect to F on  $\Lambda$ .

**Proof.** Let  $\lambda^* \in \bar{F} \cap \Lambda$  and define the set  $O_{\lambda^*} = \{g\lambda^*: g \in G\}$ . Let  $Q(x, g\lambda^*) \stackrel{\text{def}}{=} f(g^{-1}x)$  and note that Q is G-ordered on  $X \times O_{\lambda^*}$ . To see this, note that when  $g = I_n$ , Q is G-ordered with respect to F and that  $Q(g_1 x, g_1 g\lambda^*) =$



$f((g_1 g)^{-1} g_1 x) = f(g^{-1} g_1^{-1} g_1 x) = f(g^{-1} x) = Q(x, g\lambda^*)$ . By Theorem 3.11,  $Q^*(\lambda, g\lambda^*) \stackrel{\text{def}}{=} \int K(\lambda, x) Q(x, g\lambda^*) d\mu(x)$  is G-ordered on  $\Lambda \times 0_{\lambda^*}$ . But for  $g = I_n$ ,  $Q^*(\lambda, I_n \lambda^*) = h(\lambda)$ . Thus  $h(\lambda)$  is G-ordered with respect to F on  $\Lambda$ , as desired. ||

Corollary 3.13. Let  $f_1$  and  $f_2$  be G-ordered with respect to F functions. Define  $f(g) = \sum_{g_0 \in G} f_1(g^{-1} g_0) f_2(g_0)$ . Then f is G-ordered with respect to F.

#### 4. G-majorization, G-monotonicity, and the Preservation Theorem.

G-majorization is a partial ordering on  $R^n$  introduced by Eaton and Perlman (1976) (EP (1976)). G-monotone functions are isotonic with respect to this ordering. In this section we relate reflection ordering to G-majorization and show that G-monotone functions are a special case of G-ordered functions. We use the properties of G-ordered functions to obtain a preservation theorem for G-monotone functions under an integral transform. We supply a brief summary of relevant parts of the work of Eaton and Perlman.

Definition 4.1. Let  $x, y \in R^n$  and let  $x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[n]}$  and  $y_{[1]} \geq y_{[2]} \geq \dots \geq y_{[n]}$  be decreasing rearrangements of the elements of x and y respectively. Then x is said to majorize y ( $x \geq^m y$ ) if  $\sum_{i=1}^k x_{[i]} \geq \sum_{i=1}^k y_{[i]}$ ,  $k = 1, 2, \dots, n-1$ , with equality when  $k = n$ .

Definition 4.2. A function f defined on  $R^n$  is Schur-convex (Schur-concave) if  $x \geq^m y$  implies  $f(x) \geq (\leq) f(y)$ .

Majorization induces a partial ordering on  $R^n$  and Schur-convex functions are order preserving with respect to majorization. The G-majorization ordering of EP (1976) includes majorization as a special case.

Definition 4.3. Let  $x, y \in X$  and let  $G$  be a closed subgroup of  $O(n)$ . The point  $x$  is said to G-majorize  $y$  ( $x \geq^G y$ ) if  $y$  is an element of the convex hull of the  $G$ -orbit of  $x$ .

The term "G-majorization" and the notation " $x \geq^G y$ " are ours, but as mentioned before, the concept of the ordering belongs to Eaton and Perlman. When  $G = P_n$ , the group of permutation matrices, G-majorization coincides with the familiar majorization ordering of Definition 4.1. (See EP(1976).)

Definition 4.4. An extended real valued function  $f$  on  $X$  is G-monotone increasing (decreasing) if  $x \geq^G y$  implies  $f(x) \geq$  ( $\leq$ )  $f(y)$ .

When  $G = P_n$ , the group of permutation matrices, the class of G-monotone increasing (decreasing) functions coincides with the class of Schur-convex (Schur-concave) functions.

Lemma 4.5. (EP(1976)). Let  $G$  be a finite reflection group. Suppose  $x \geq^G y$ ,  $x \neq y$ . Then there exists a sequence of points  $z_0, z_1, \dots, z_m$  such that  $z_0 = y$ ,  $z_m = x$ , and  $z_{j-1} = [\lambda_j I_n + (1-\lambda_j) M_{r_j}] z_j$ ,  $1 \leq j \leq m$ , where  $r_j \in \Delta_G$ ,  $0 \leq \lambda_j < 1$ , and  $I_n$  is the  $n \times n$  identity matrix.

Note that  $z_j \geq^G z_{j-1}$  for all  $j$ .

Eaton and Perlman devote a significant part of their paper to establishing the above path lemma. It demonstrates the existence of a polygonal path between a point  $x \in R^n$  and any point  $y$  in the convex hull of the  $G$ -orbit of  $x$ .

This lemma is a key tool in the study of G-majorization and G-monotone functions. For example, to show that a function  $f$  on  $R^n$  is G-monotone increasing it suffices to show that for all  $x \in R^n$  and  $r \in \Delta_G$ ,  $f(x) \geq f([\lambda I_n + (1-\lambda) M_r]x)$ . The above procedure is only one method for determining G-monotone functions. We refer the reader to EP(1976) for their differential characterizations of smooth G-monotone functions.

Before we show the relationship between G-ordered functions and G-monotone functions we need to establish some technical lemmas. We will find the following notation useful in the remainder of the paper.

Lemma 4.6. Let  $r \in \Delta_G$  and let  $u_1, u_2, \dots, u_n$  be an orthonormal basis for  $R^n$  such that  $u_1 = r$ . Suppose  $x, y \in R^n$  are such that  $u_i' x = u_i' y$ ,  $i = 2, 3, \dots, n$ . Then  $x \geq^G y$  if and only if  $|r'x| \geq |r'y|$ .

Proof. The point  $y$  is on the line which passes through the points  $x$  and  $M_r x$ . Consequently,  $y$  is in the convex hull of the  $G$ -orbit of  $x$  if and only if  $|r'x| \geq |r'y|$ . ||

**Lemma 4.7.** Let  $F$  be a fundamental region in  $R^n$ . Then  $g \geq^F M_r g$  if and only if  $(r'gx) \geq 0$  for all  $x \in \bar{F}$ .

**Proof.** By definition,  $g \geq^F M_r g$  if and only if  $g \in G_r^+$ . But  $g \in G_r^+$  if and only if  $g \bar{F} \subset H_r^+ \cup H_r^0$ . Now  $gx \in H_r^+ \cup H_r^0$  if and only if  $(r'gx) \geq 0$  and  $g\bar{F} \subset H_r^+ \cup H_r^0$  if and only if  $gx \in H_r^+ \cup H_r^0$  for all  $x \in \bar{F}$ . Thus the result is immediate. ||

**Lemma 4.8.** Let  $r \in \Delta_G$  and let  $x \in \bar{F}$ . Then  $g \geq^{\bar{F}} M_r g$  if and only if  $\lambda + gx \geq^G \lambda + M_r gx$  ( $\lambda - M_r gx \geq \lambda - gx$ ) for all  $\lambda \in H_r^+ \cup H_r^0$ .

**Proof.** We will show that  $g \geq^{\bar{F}} M_r g$  if and only if  $\lambda + gx \geq^G \lambda + M_r gx$  for all  $\lambda \in H_r^+ \cup H_r^0$ . The proof that  $g \geq^{\bar{F}} M_r g$  if and only if  $\lambda - M_r gx \geq^G \lambda - gx$  is analogous. Without loss we assume  $g = I_n$  and we show that  $\lambda + x \geq^G \lambda + M_r x$  if and only if  $(r'\lambda)(r'x) \geq 0$ . Let  $u_1, u_2, \dots, u_n$  be an orthonormal basis for  $R^n$  such that  $u_1 = r$ . Now  $\lambda + x = ((r'\lambda + r'x)r + \sum_{i=2}^n (u_i'\lambda + u_i'x)u_i)$  and  $\lambda + M_r x = ((r'\lambda - r'x)r + \sum_{i=2}^n (u_i'\lambda + u_i'x)u_i)$ . So  $\lambda + x \geq^G \lambda + M_r x$  if and only if  $|r'\lambda + r'x| \geq |r'\lambda - r'x|$  as a consequence of Lemma 4.6. But  $|r'\lambda + r'x| \geq |r'\lambda - r'x|$  if and only if  $(r'\lambda)(r'x) \geq 0$ . Thus  $\lambda + x \geq^G \lambda + M_r x$  if and only if  $(r'\lambda)(r'x) \geq 0$ . By Lemma 4.7,  $g \geq^F M_r g$  if and only if  $(r'\lambda)(r'gx) \geq 0$  for all  $\lambda \in H_r^+ \cup H_r^0$  and all  $x \in \bar{F}$ . In particular,  $I_n \geq^{\bar{F}} M_r I_n$  if and only if  $(r'\lambda)(r'x) \geq 0$  for all  $\lambda \in H_r^+ \cup H_r^0$ . Thus  $g \geq^{\bar{F}} M_r g$  if and only if  $\lambda + gx \geq^G \lambda + M_r gx$  for all  $\lambda \in H_r^+ \cup H_r^0$ . ||

**Lemma 4.9.** Let  $r \in S_n$  and let  $z \in R^n$ . Then for any  $\alpha$ ,  $0 \leq \alpha < 1$ , there exist  $\lambda_\alpha, x_\alpha \in R^n$  such that  $z = \lambda_\alpha + x_\alpha$ ,  $(r'\lambda_\alpha)(r'x_\alpha) \geq 0$ , and  $(\alpha I_n + (1 - \alpha)M_r)(\lambda_\alpha + x_\alpha) = \lambda_\alpha + M_r x_\alpha$ .



Proof. Let  $x_\alpha = (1 - \alpha)z$  and  $\lambda_\alpha = \frac{\alpha}{1-\alpha} x_\alpha$ , then

- (i)  $\lambda_\alpha + x_\alpha = \frac{\alpha}{1-\alpha} x_\alpha + x_\alpha = \frac{1}{1-\alpha} x_\alpha = z$
- (ii)  $(r'\lambda_\alpha)(r'x_\alpha) = (r'\frac{\alpha}{1-\alpha} x_\alpha)(r'x_\alpha) = \frac{\alpha}{1-\alpha} (r'x_\alpha)^2 \geq 0.$
- (iii)  $(\alpha I_n + (1 - \alpha)M_r)(\lambda_\alpha + x_\alpha) = \frac{\alpha}{1-\alpha} x_\alpha + M_r x_\alpha = \lambda_\alpha + M_r x_\alpha. \quad ||$

Theorem 4.10. Let  $K(\lambda, x)$  be of the form  $f(\lambda + x)(f(\lambda - x))$ . Then  $K(\lambda, x)$  is G-ordered on  $R^{2n}$  if and only if  $f(\lambda + x)(f(\lambda - x))$  is G-monotone increasing (decreasing) on  $R^n$ .

Proof. We show that  $K(\lambda, x)$  is G-ordered if and only if  $f(\lambda + x)$  is G-monotone increasing. The proof that  $K(\lambda, x)$  is G-ordered if and only if  $f(\lambda - x)$  is G-monotone decreasing is analogous.

(i) We show for all  $g \in G$ ,  $K(\lambda, x) = K(g\lambda, gx)$  if and only if  $f(\lambda + x) = f(g(\lambda + x))$ . Now  $K(\lambda, x) = f(\lambda + x)$  and  $K(g\lambda, gx) = f(g\lambda + gx) = f(g(\lambda + x))$ . So  $K(\lambda, x) = K(g\lambda, gx)$  if and only if  $f(\lambda + x) = f(g(\lambda + x))$ .

(iia) Suppose  $f$  is G-monotone increasing. Let  $\lambda \in \bar{F} \cap \Lambda$ ,  $x \in \bar{F} \cap X$ , and  $r \in \Delta_t^+$  for every  $t \in F$ . Then  $(r'\lambda)(r'x) \geq 0$  which implies  $\lambda + x \geq^G \lambda + M_r x$  by Lemmas 4.7 and 4.8. Consequently  $K(\lambda, x) - K(\lambda, M_r x) = f(\lambda + x) - f(\lambda + M_r x) \geq 0$ .

(iib) Suppose  $K$  is G-ordered on  $\Lambda \times X$  and let  $z_1 \geq^G z_2$ . Without loss assume  $z_2 = (\alpha I_n + (1 - \alpha)M_r)z_1$  for any arbitrary  $\alpha$ ,  $0 \leq \alpha < 1$ . By Lemma 4.9, there exist  $\lambda_\alpha, x_\alpha$  such that  $z_1 = \lambda_\alpha + x_\alpha$ ,  $z_2 = \lambda_\alpha + M_r x_\alpha$ , and  $(r'\lambda_\alpha)(r'x_\alpha) \geq 0$ . Consequently  $f(z_1) - f(z_2) = f(\lambda_\alpha + x_\alpha) - f(\lambda_\alpha + M_r x_\alpha) = K(\lambda_\alpha, x_\alpha) - K(\lambda_\alpha, M_r x_\alpha) \geq 0$  since  $(r'\lambda_\alpha)(r'x_\alpha) \geq 0$  implies  $I_n \geq^{\bar{F}} M_r I_n$  by Lemma 4.7.  $||$

Definition 4.11. A measure  $\mu$  on  $X$  is said to be translation invariant if  $\lambda(A \cap X) = \mu((A + x) \cap X)$  for all Borel sets  $A \subseteq \mathbb{R}^n$  and all  $x \in X$ .

Corollary 4.12. The convolution of  $G$ -monotone decreasing functions on  $\mathbb{R}^n$  with respect to a translation and  $G$ -invariant measure  $\mu$  is  $G$ -monotone decreasing.

Proof. Let  $f_1, f_2$  be  $G$ -monotone decreasing on  $\mathbb{R}^n$  and define  $h(x) = \int f_1(x - y) f_2(y) d\mu(y)$ . Then  $h(x - z) = \int f_1(x - z - y) f_2(y) d\mu(y) = \int f_1(x - u) f_2(u - z) d\mu(u)$ , using the translation  $u = z + y$ . By Theorem 4.10,  $f_1(x - u), f_2(u - z)$  are  $G$ -ordered on  $\mathbb{R}^{2n}$ . By Theorem 3.11,  $h(x - z)$  is  $G$ -ordered on  $\mathbb{R}^{2n}$ . Thus by another application of Theorem 4.10,  $h$  is  $G$ -monotone decreasing on  $\mathbb{R}^n$ . ||

The convolution theorem (Theorem 5.1) of Eaton and Perlman (1976) is a stronger result than Corollary 4.12. They prove that the convolution of  $G$ -monotone decreasing functions is  $G$ -monotone decreasing for a general, not necessarily finite, reflection group.

Remark. For Corollary 4.12 it is not necessary that the functions be  $G$ -monotone decreasing on  $\mathbb{R}^n$ . Suppose  $X$  is a subset of  $\mathbb{R}^n$  such that the set  $U \stackrel{\text{def}}{=} \{u \in \mathbb{R}^n: u = x + y, x, y \in X\}$  is  $G$ -invariant, then the convolution of  $G$ -monotone decreasing functions on  $X$  is  $G$ -monotone decreasing. The reader should note that the above condition is satisfied if  $X$  forms a semigroup under addition, for then  $U \equiv X$ .

Definition 4.13. Let  $\Lambda, X \subseteq \mathbb{R}^n$  form semigroups under addition. A function  $K(\lambda, x)$  on  $\Lambda \times X$  is said to have the  $G$ -ordered generalized semigroup property with respect to a translation invariant measure  $\mu$  if, for  $\lambda_1, \lambda_2 \in \Lambda$  and  $x \in X$ , there exist



G-ordered functions  $K^{(1)}(\lambda, x)$  and  $K^{(2)}(\lambda, x)$  on  $\Lambda \times X$  such that  $K(\lambda_1 + \lambda_2, x) = \int K^{(1)}(\lambda_1, x - y) K^{(2)}(\lambda_2, y) d\mu(y)$ .

**Theorem 4.14.** Let  $\Lambda, X$  be as in Definition 4.13 and let a function  $K$  on  $\Lambda \times X$  have the G-ordered generalized semigroup property with respect to a G-invariant and translation invariant measure  $\mu$ . Then  $h(\lambda) = \int K(\lambda, x) f(x) d\mu(x)$  is G-monotone increasing (decreasing) on  $\Lambda$  if  $f(x)$  is G-monotone increasing (decreasing) on  $X$ .

**Proof.** We show that  $F$  is G-monotone increasing implies that  $h$  is G-monotone increasing. We will show that  $h(\lambda + \lambda^*)$  is G-ordered on  $\Lambda^2$  and conclude that  $h$  is G-monotone increasing on  $\Lambda$  using Theorem 4.10. Write

$$\begin{aligned} h(\lambda + \lambda^*) &= \int_X K(\lambda + \lambda^*, x) f(x) d\mu(x) \\ &= \int_X \int_X K^{(1)}(\lambda, x - y) K^{(2)}(\lambda^*, y) d\mu(y) f(x) d\mu(x) \\ &= \int_X K^{(2)}(\lambda^*, y) \int_X K(\lambda, x - y) f(x) d\mu(x) d\mu(y) \\ &= \int_X K^{(2)}(\lambda^*, y) \int_{X_y} K^{(1)}(\lambda, z) f(y + z) d\mu(z) d\mu(y), \end{aligned}$$

where  $X_y = \{u \in \mathbb{R}^n: u = x - y, x, y \in X\}$ . [We use the translation  $z = x - y$ .]

Since  $X$  forms a semigroup under addition,  $X_y \supseteq X$  for all  $y \in X$ . On the set

$X_y - X$ ,  $K^{(1)}(\lambda, \cdot)$  is zero, so we can replace  $X_y$  by  $X$  for the region of

integration of the inside integral. Thus  $h(\lambda + \lambda^*) = \int_X K^{(2)}(\lambda^*, y)$

$\int_X K^{(1)}(\lambda, z) f(y + z) d\mu(z) d\mu(y)$ . By an application of Theorem 2.12,

$\int_X K^{(1)}(\lambda, z) f(y + z) d\mu(z)$  is G-ordered. By a second application of Theorem

2.12,  $h(\lambda + \lambda^*)$  is G-ordered on  $\Lambda^2$ . Thus  $h$  is G-monotone increasing on  $\Lambda$ . To

show  $f$  G-monotone decreasing implies  $h$  G-monotone decreasing, we need only

consider  $-f$  which is G-monotone increasing and deduce that  $-h$  is G-monotone increasing.

We give a simple example to illustrate the concepts and notation. The reader will see how Theorem 4.14 can be used to develop monotonicity properties for power functions in hypothesis testing. Applications of significance are given in Part II.

Example 4.15. Consider the group  $G$  generated by  $\Delta_G^* = \left\{ \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -\sqrt{2} & \sqrt{2} \end{pmatrix} \right\}$ .

Let  $K(\lambda, x)$  be the density of an exchangeable bivariate normal random variable. It can be verified that  $K$  has the  $G$ -ordered generalized semigroup property. If we wish to test the hypothesis  $H_0: \lambda = 0$  against the alternative  $H_1: \lambda \neq 0$ , then a test with critical region  $Q_f = \{x \in \mathbb{R}^2: f(x) > c\}$  where  $f$  is a  $G$ -monotone increasing function has a  $G$ -monotone increasing power function. As an example, consider  $f(x) = |x_1| + |x_2|$ . The power function,  $h(\lambda) = \int K(\lambda, x) I_{Q_f}(x) dx$ , is  $G$ -monotone increasing since  $I_{Q_f}(x)$ , the indicator function, is  $G$ -monotone increasing. Thus any test of  $H_0$  versus  $H_1$  with critical region  $Q_f$  is unbiased and has a power function which increases monotonically as either or both of the coordinates of  $\lambda$  increase in absolute value.

Theorem 4.14 is an extension and generalization of a similar preservation theorem under an integral transform (Theorem 3.7) of Hollander, Proschan, and Sethuraman (to appear). It yields their theorem as a special case when  $G = P_n$ , the group of permutation matrices,  $K^{(1)} = K^{(2)} = K$ , and  $\Lambda$  and  $X$  are the positive reals or the positive integers.

Definition 4.16. Let  $\Lambda, X$  be as in Definition 4.13. A function  $K(\lambda, x)$  on  $\Lambda \times X$  is said to have the  $G$ -ordered conditional generalized semigroup property with respect to a translation invariant measure  $\mu$ , if there exists a  $\sigma$ -finite measure space

$(\Omega, \mathcal{F}, \nu)$  and functions  $K_\omega(\lambda, x)$ ,  $\omega \in \Omega$ , such that:

$$i. \quad K(\lambda, x) = \int_{\Omega} K_\omega(\lambda, x) \, d\nu(\omega).$$

ii. For each  $\omega \in \Omega$ ,  $K_\omega$  has the G-ordered generalized semigroup property with respect to  $\mu$ .

Corollary 4.17. The conclusion of Theorem 4.14 holds if  $K(\lambda, x)$  has the G-ordered conditional generalized semigroup property.

Proof. Let  $h_\omega(\lambda) = \int_X K_\omega(\lambda, x) f(x) \, d\mu(x)$ . Then by Theorem 4.14,  $h_\omega(\lambda)$  is G-monotone increasing (decreasing) on  $\Lambda$  for each  $\omega \in \Omega$ . Now

$$\begin{aligned} h(\lambda) &= \int_X K(\lambda, x) f(x) \, d\mu(x) \\ &= \int_X \int_{\Omega} K_\omega(\lambda, x) \, d\nu(\omega) f(x) \, d\mu(x) \\ &= \int_{\Omega} \int_X K_\omega(\lambda, x) f(x) \, d\mu(x) \, d\nu(\omega) \\ &= \int_{\Omega} h_\omega(\lambda) \, d\nu(\omega). \end{aligned}$$

By the mixture result, Proposition 3.8, and Theorem 4.10 we conclude that  $h(\lambda)$  is G-monotone increasing (decreasing). ||

It should be noted that in the case where  $G = P_n$  we have relaxed the assumptions imposed on  $K(\lambda, x)$ , the kernel of the transform, by Hollander, Proschan, and Sethuraman (to appear). We conclude that all the results relating to statistical applications of their preservation theorem under an integral transform are applicable to a much larger class of probability distributions, in particular, to the distributions of exchangeable normal and  $t$  random variables.



Example 4.18. A bivariate  $T$  random variable can be expressed as  $T = \frac{\sqrt{n}}{\bar{X}} U$ , where  $X^2$  is a chi-square random variable with  $n$  degrees of freedom and  $U$  is an exchangeable bivariate normal variable. If  $\lambda$  is the mean of  $U$ , then  $K(\lambda, x)$ , the density of  $T$ , has the  $G$ -ordered conditional generalized semigroup property. Thus for the same test, with critical region  $Q_f$ , the power function has the monotonicity property described in Example 4.15.

### 5. A Sample Application.

In this section we consider an application in the area of nonparametric statistics. We show that the Wilcoxon signed-rank statistic is a function  $G$ -ordered with respect to  $F$  of the signed ranks. Using results from previous sections we will determine under what conditions the frequency function of the signed ranks is  $G$ -ordered. We also present monotonicity properties for power functions of tests based on the Wilcoxon signed-rank statistic.

Let  $G$  be the group of all permutations and sign changes of the coordinates acting on the elements of  $R^n$ . Given a set of real numbers  $\{x_1, x_2, \dots, x_n\}$ , let  $r_i = 1 + \sum_{j \neq i} L(|x_i|, |x_j|)$ ,  $i = 1, 2, \dots, n$ , where  $L(a, b) = 1$  if  $a > b$ ,  $\frac{1}{2}$  if  $a = b$ , and  $0$  if  $a < b$ . Let  $v_i = (\text{sgn } x_i)r_i$ , then  $v = (v_1, v_2, \dots, v_n)$  is the vector of signed ranks of the set of real numbers  $\{x_1, x_2, \dots, x_n\}$ . Define the function  $J(x, v)$  on  $R^{2n}$  as follows:

$$J(x, v) = \begin{cases} 1 & \text{if } v_i = (\text{sgn } x_i)r_i, i = 1, 2, \dots, n \\ 0 & \text{otherwise.} \end{cases}$$

It can be verified that  $J(x, v)$  is  $G$ -ordered for the group  $G$  of permutations and sign changes. The frequency function  $g(\lambda, v)$  of the signed ranks  $v$  can be written as follows:

$$g(\lambda, v) = \int K(\lambda, x) J(x, v) d\mu(x).$$

If  $K(\lambda, x)$  is  $G$ -ordered and  $\mu$  is a  $G$ -invariant measure, then by the composition theorem (Theorem 3.11),  $g(\lambda, v)$  is  $G$ -ordered. We note that both Lebesgue measure and counting measure are  $G$ -invariant for the group of permutations and sign changes.

Define the function  $f(x)$  on  $R^n$  in the following manner:

$$f(x) = \sum_{i=1}^n x_i I(x_i > 0).$$

Then  $f$  is  $G$ -ordered with respect to  $F$  with  $F = \{x \in R^n: x_1 < x_2 < \dots < x_n\}$ . Let  $Q_f = \{x \in R^n: f(v) > c_\alpha\}$ . Since  $f$  is  $G$ -ordered with respect to  $F$ , the indicator function  $I(x \in Q_f)$  is  $G$ -ordered with respect to  $F$ . If  $K(\lambda, x)$  is  $G$ -ordered, then by the composition theorem for functions  $G$ -ordered with respect to  $F$  (Corollary 3.12), the power function,  $h(\lambda) \stackrel{\text{def}}{=} \int K(\lambda, x) I(x \in Q_f) d\mu(x)$ , of any test based upon the Wilcoxon signed-rank statistic is a function  $G$ -ordered with respect to  $F$  on the parameter space.

Actually a stronger result than this is true. Define the power function  $h(\lambda)$  as  $h(\lambda) = \int g(\lambda, v) I(f(v) > c_\alpha) d\mu(v)$ . By Corollary 3.12,  $h(\lambda)$  is  $G$ -ordered with respect to  $F$  if  $g(\lambda, v)$  is  $G$ -ordered. Thus it is not necessary that  $K(\lambda, x)$  be  $G$ -ordered. It suffices that the frequency function  $g(\lambda, v)$  of the signed ranks be  $G$ -ordered.

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This is Part I of a two-part paper which generalizes and extends the work of Hollander, Proschan, and Sethuraman (to appear) for functions decreasing in transposition (IT). It In Part I we introduce reflection ordering and G-ordered functions, and develop many preservation properties for these functions. We note that G-ordered functions contain, as a special case, G-monotone functions (Eaton and Periman (1976)), and we prove a preservation theorem for G-monotone functions under an integral transform. We extend the results of Hollander, Proschan, and Sethuraman (to appear), in the area of stochastic are extended			

## 20. ABSTRACT CONTINUED

→ comparisons to a larger class of probability distributions. In Part II we present applications in statistics. In a forthcoming paper we will explore the notion of stochastic G-majorization as an extension of the work done in the area of stochastic majorization by Nevius, Proschan, and Sethuraman (1977).